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EXOTIC NEGATIVELY CURVED STRUCTURES ON CAYLEY HYPERBOLIC MANIFOLDS

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Abstract

We construct examples of closed negatively curved manifolds M which are homeomorphic but not diffeomorphic to Cayley locally symmetric spaces. Given $\epsilon > 0$, we can construct such an M with sectional curvatures all in $[-4-\epsilon, -1]$.

1. Introduction

Margulis [16] discovered a strengthening of Mostow's strong rigidity theorem [17] to a phenomenon called Archimedean superrigidity valid for lattices in semisimple Lie groups G of real rank bigger than or equal to two. (Here G is assumed to be centerless and to contain no compact normal subgroup other than 1.) Later Corlette [5] proved a version of superrigidity for lattices in the automorphism groups of quarternion hyperbolic spaces or the Cayley hyperbolic plane. It is known that superrigidity fails for other real rank 1 situations; i.e., for lattices in the automorphism groups of the real or complex hyperbolic spaces. Stronger versions of Corlette superrigidity were later proven by Jost and Yau [13] and Mok, Siu and Yeung [19]. A consequence of these superrigidity theorems is that if M and N are homeomorphic closed negatively curved manifolds and the universal cover of M is either a quaternionic hyperbolic space $\mathbb{H}\mathbf{H}^n$, $n \geq 2$, or the Cayley hyperbolic plane $\mathbb{O}\mathbf{H}^2$, then M and N are isometric up to a scaling of the metric on either of them by a

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constant (the isometry being the unique harmonic map in the homotopy class of the homeomorphism [6], [10]) under any of the following three extra conditions on N:

1. The curvature operator of N is nonpositive [5].
2. The complexified sectional curvatures of N are nonpositive [19].
3. The sectional curvatures of N are pointwise $\frac{1}{4}$-pinched; i.e., lie in a closed interval $[-4a_x, -a_x]$ where $a_x > 0$ and $x \in N$ (cf [11] and [24]).

In fact, each of Conditions 1 and 3 independently imply Condition 2.

In [9], homeomorphic pairs of closed negatively curved n-manifolds M and N are constructed where the universal cover $\widetilde{M}$ of M is the complex hyperbolic space $\mathbb{C}\mathbf{H}^m$ (and $n = 2m$) but M and N are not diffeomorphic; indeed, given $\epsilon > 0$, such pairs of M and N were constructed so that $\widetilde{M} = \mathbb{C}\mathbf{H}^m$ and the sectional curvatures of N are "almost $\frac{1}{4}$-pinched", i.e., lie in $[-4-\epsilon, -1]$.

It was conjectured in [9] that such examples could be constructed where the universal cover $\widetilde{M}$ of M is either the quaternionic hyperbolic plane $\mathbb{H}\mathbf{H}^2$ or $\mathbb{O}\mathbf{H}^2$. We prove here this conjecture for the case where $\widetilde{M} = \mathbb{O}\mathbf{H}^2$. The smooth manifolds N are the connected sum $M\#\Sigma^{16}$ where Σ^{16} is the unique smooth manifold homeomorphic but not diffeomorphic to the 16-dimensional round sphere S^{16}. The case when $\widetilde{M} = \mathbb{H}\mathbf{H}^n$ is treated separately in [2] where we use a different technique to show that the manifolds $M\#\Sigma^{4n}$ admit metrics of negative curvature but get a weaker result without the "almost 1/4-pinched" conclusion. However, we believe that the method used in this paper could be used to get the pinching result for the case $\widetilde{M} = \mathbb{H}\mathbf{H}^2$. A corollary of our construction is that Condition 1 or 2 on N in the superrigidity theorems mentioned above is optimal in a sense, i.e., neither of them can be replaced by the condition that the sectional curvatures of N are nonpositive. On the other hand, in view of superrigidity under Condition 3 on N, we note that ϵ cannot be 0 in any of our examples.

We conclude this introduction with an outline of the paper. There are two problems that must be addressed: (1) How to put a negatively curved metric on $M\#\Sigma^{16}$. (2) How to show that $M\#\Sigma^{16}$ is not diffeomorphic to M. ($M\#\Sigma^{16}$ is clearly homeomorphic to M.) We solve problem (1) in the next section and problem (2) in the final section of the paper. Broadly speaking, we follow the pattern established in

[8] and [9]. But the difficulties encountered are more formidable and require substantial modifications to the arguments in [9].

To solve the first problem, we construct a 1-parameter family $b_\gamma(\ ,\ )$ of Riemannian metrics on $\mathbb{R}^{16}$ indexed by $\gamma \in [e, +\infty)$ which satisfy the following properties:

(i) The sectional curvatures of b_γ lie in the closed interval $[-4 - \epsilon(\gamma), -1]$ where $\epsilon(\gamma) > 0$ and $\epsilon(\gamma) \to 0$ as $\gamma \to +\infty$.

(ii) The ball of radius γ about 0 in $(\mathbb{R}^{16}, b_\gamma)$ is isometric to a ball of radius γ in real hyperbolic space $\mathbb{R}\mathbf{H}^{16}$.

(iii) The complement of the ball of radius γ^2 about 0 in $(\mathbb{R}^{16}, b_\gamma)$ is isometric to the complement of a ball of radius γ^2 in $\mathbb{O}\mathbf{H}^2$.

To construct these metrics we make use of the explicit description of the Riemannian curvature tensor for $\mathbb{O}\mathbf{H}^2$ given in [4]. We use this result together with [9, Lemma 3.18] to put an "almost $\frac{1}{4}$-pinched" negatively curved Riemannian metric on $M\#\Sigma^{16}$ provided M has sufficiently large injectivity radius. Here M is a closed, orientable Cayley hyperbolic manifold. This injectivity radius condition is satisfied when we pass to sufficiently large finite sheeted covers of M since $\pi_1(M)$ is a residually finite group.

The second problem (i.e., to show that M and $M\#\Sigma^{16}$ are not diffeomorphic) is reduced via Kirby-Siebenmann smoothing theory and using Mostow's strong rigidity theorem [17] together with its topological analogue [7] to showing that the group homomorphism

$$\theta_{16} = [S^{16}, \mathrm{Top}/O] \underset{\phi^*}{\longrightarrow} [M, \mathrm{Top}/O]$$

is monic where $\phi : M \to S^{16}$ is a degree 1 map. Now, a result of Okun [21] shows that ϕ^* is the initial map in a factoring of

$$\theta_{16} = [S^{16}, \mathrm{Top}/O] \underset{\psi^*}{\longrightarrow} [\mathbb{O}\mathbf{P}^2, \mathrm{Top}/O]$$

where $\psi : \mathbb{O}\mathbf{P}^2 \to S^{16}$ is a degree 1 map and $\mathbb{O}\mathbf{P}^2$ is the Cayley projective plane. Hence it suffices to show that ψ^* is monic. This is done by making delicate use of some calculations of Toda [22] on the stable homotopy groups of spheres.

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2. Tapering between $\mathbb{O}\mathbf{H}^2$ and $\mathbb{R}\mathbf{H}^{16}$

We begin with a brief description of the Cayley hyperbolic plane $\mathbb{O}\mathbf{H}^2$.

The Cayley numbers, denoted by $\mathbb{O}$, is an 8-dimensional non associative division algebra over the real numbers. It has a multiplicative identity 1 and a positive definite bilinear form $\langle\ ,\ \rangle$ whose associated norm $\|\ \|$ is multiplicative, i.e., $\|uv\| = \|u\|\|v\|$. Every element $u \in \mathbb{O}$ can be written as $\alpha + u_0$ when α is real and $\langle \alpha, u_0 \rangle = 0$. The conjugation map $u \mapsto \bar{u} := \alpha - u_0$ is an antiautomorphism, i.e., $\overline{(uv)} = \bar{v}\bar{u}$ for all $u, v \in \mathbb{O}$. Moreover, $u\bar{u} = \|u\|^2$ and one has the following identities which can be checked easily: $\langle uv, w \rangle = \langle \bar{v}\bar{u}, \bar{w} \rangle = \langle \bar{v}, \bar{w}u \rangle = \langle v, \bar{u}w \rangle$ for $u, v, w \in \mathbb{O}$.

On $\mathbb{O}^2 = \mathbb{O} \times \mathbb{O}$, one has the positive definite bilinear form given by $\langle (u_1, u_2), (v_1, v_2) \rangle = \langle u_1, v_1 \rangle + \langle u_2, v_2 \rangle$ for $u_1, v_1, u_2, v_2 \in \mathbb{O}$. The set $D = \{u \in \mathbb{O}^2 \mid \langle u, u \rangle < 1\}$ equipped with the metric given by formula (20.4) in [17, p. 144] is a model for the Cayley hyperbolic plane.

It is convenient for us to consider the set $\mathbb{O}^2$ itself as the underlying set for the Cayley hyperbolic plane $\mathbb{O}\mathbf{H}^2$ equipped with the metric gotten by scaling the above metric from D to $\mathbb{O}^2$. This enables one to identify $\mathbb{O}^2$ with $T_0(\mathbb{O}\mathbf{H}^2)$, the tangent space to $\mathbb{O}\mathbf{H}^2$ at the origin $0 \in \mathbb{O}^2$.

The Riemannian metric on the distance sphere S^{15} at distance t from the origin can be described as follows. Firstly, one has the Hopf fibering of S^{15} over S^8 with S^7 as fiber. This equips S^{15} with complementary distributions η_1, η_2 where Whitney sum $\eta_1 \oplus \eta_2$ equals the tangent bundle of S^{15}. η_1 is the 7-dimensional distribution tangent to the S^7 fibers and η_2 is the 8-dimensional distribution perpendicular to η_1 (perpendicular with respect to the round metric on S^{15}). We call the subspace of the tangent space to S^{15} belonging to the distribution η_1 "the vertical subspace" and the subspace belonging to η_2 "the horizontal subspace". The induced Riemannian metric $\langle\ ,\ \rangle$ on S^{15} is then,

$$\langle X, X \rangle = b^2 X \cdot X, \quad \langle U, U \rangle = a^2 U \cdot U \quad \text{and} \quad \langle X, U \rangle = 0$$

where $X \in \eta_1$, $U \in \eta_2$, $a = \sinh t$, $b = \sinh t \cosh t$ and "$\cdot$" is the inner product with respect to the round metric on S^{15}. For brevity, we denote the distance sphere with the above metric by $S^{15}_{a,b}$.

Fix a smooth function $\phi : (0, +\infty) \to [0, +\infty)$ such that $\dot{\phi}(t) \geq 0$ for all $t \in [0, +\infty)$. We put a Riemannian metric on $S^{15} \times (0, +\infty)$ using the function $\phi(t)$ as follows. The foliations $S^{15} \times t$ and $x \times (0, +\infty)$ are required to be perpendicular where $x \in S^{15}$ and $t \in (0, +\infty)$. We set $|N| = 1$ where $N = \dfrac{\partial}{\partial t}$ and t is the second coordinate variable in the product structure $S^{15} \times (0, +\infty)$. We require that the induced metric on $S^{15} \times t$ is $S^{15}_{a,b}$ where $a = \sinh t$ and $b = \sinh t \cosh \phi(t)$. This Riemannian manifold is denoted $S^{15} \times^{\phi} (0, +\infty)$. Notice that when $\phi(t) = t$ for all $t \in (0, +\infty)$, $S^{15} \times^{\phi} (0, +\infty)$ is the punctured Cayley hyperbolic plane $\mathbb{O}\mathbf{H}^2 - *$. Our object is to calculate the sectional curvatures of $S^{15} \times^{\phi} (0, +\infty)$.

Let P be a real 2-plane tangent to $S^{15} \times^{\phi} (0, +\infty)$. Let τ denote the angle made by the plane P with the distance sphere S^{15}. Then P is spanned by vectors $\{u, \cos(\tau)v + \sin(\tau)N\}$ where vectors $\{u, v\}$ are tangent to the $S^{15} \times t$ foliation and satisfy $|u| = |v| = 1$ and $(u \cdot v) = 0$ where $| \, |$ denotes the norm and $(\cdot)$ denotes the inner product with respect to the metric on $S^{15} \times^{\phi} (0, +\infty)$. If $\overline{K}(P)$ denotes the sectional curvature of the plane P in $S^{15} \times^{\phi} (0, +\infty)$, then we have

$$
\begin{aligned}
(1) \qquad \overline{K}(P) &= \overline{K}(u, \cos \tau v + \sin \tau N) \\
&= \cos^2 \tau \overline{K}(u, v) + \sin^2 \tau \overline{K}(u, N) \\
&\quad + 2 \sin \tau \cos \tau (\overline{R}(u, N)v \cdot u)
\end{aligned}
$$

where $\overline{R}$ denotes the Riemann curvature tensor in $S^{15} \times^{\phi} (0, +\infty)$.

We shall denote the vectors tangent to S^{15} and lying in the vertical subspace by symbols X, Y and those lying in the horizontal subspace by symbols U, V. If σ is the angle between the vector u and the horizontal subspace and α is the angle between v and the horizontal subspace, let $u = \sin \sigma X + \cos \sigma U$ and $v = \sin \alpha Y + \cos \alpha V$ where $|X| = |Y| = |U| = |V| = 1$. (Note $\sigma, \alpha \in [0, \pi/2]$.) We calculate $\overline{K}(P)$ by explicitly calculating $\overline{K}(u, v)$, $\overline{K}(u, N)$ and $(\overline{R}(u, N)v \cdot u)$ separately.

Before starting off to compute the above terms, we make the following important observations.

Firstly, recall that we identify $\mathbb{O}^2$ with the tangent space $T_0(\mathbb{O}\mathbf{H}^2)$ where $0 = (0, 0) \in \mathbb{O}^2$. Since $\mathbb{O}\mathbf{H}^2$ is a homogeneous space, the group G of isometries of $\mathbb{O}\mathbf{H}^2$ acts transitively on $\mathbb{O}\mathbf{H}^2$. Further, G_0 — the subgroup of G fixing 0 — acts transitively on vectors of unit length in $T_0(\mathbb{O}\mathbf{H}^2)$. Therefore one can identify the tangent spaces at other points of $\mathbb{O}\mathbf{H}^2$ also with $\mathbb{O}^2$. In particular, we identify the vector N, normal to the distance spheres, with $(1,0) \in \mathbb{O}^2$. This would identify $0 \times \mathbb{O}$ with the horizontal subspace and the subspace of $\mathbb{O} \times 0$ perpendicular to N with the vertical subspace. Thus the vectors X, Y lying in the vertical subspace are purely imaginary Cayley numbers.

Secondly, we note that any $A \in O(16)$ with the property that it induces a permutation of the fibers of the Hopf fibration $S^{15} \to S^8$ determines an isometry $\overline{A}$ of $S^{15} \times^\phi (0, +\infty)$ defined by $\overline{A}(x,t) = (A(x), t)$. All $A \in \mathrm{Spin}(9) \subseteq O(16)$ have this property. And since $\mathrm{Spin}(9)$ acts transitively on S^{15}, it acts transitively on the fibers of the Hopf fibration. In fact, for any leaf L of Hopf fibration there exists an $A \in \mathrm{Spin}(9)$ such that $A^2 = I$ and the fixed set of A (acting on S^{15}) is L. To verify this, it suffices to verify it for $L = (\mathbb{O} \times 0) \cap S^{15}$. Here we can define $A(x,y) = (x, -y)$ (cf. [4, §§3 and 4]). Consequently, the submanifolds $L \times^\phi (0, +\infty)$ are totally geodesic in $S^{15} \times^\phi (0, +\infty)$. In particular, $L \times^t (0, +\infty)$ is totally geodesic in $S^{15} \times^t (0, +\infty) = \mathbb{O}\mathbf{H}^2 - *$.

Thirdly, the Hopf submersion $S^{15} \to S^8$ is indeed a Riemannian submersion $S^{15}(1) \to S^8(r)$ where $S^{15}(1)$ is the sound sphere S^{15} of radius 1 and $S^8(r)$ is the round sphere S^8 of radius r. Therefore, one has a Riemannian submersion from $S^{15}_{a,b} \to S^8_a$ and more generally a Riemannian submersion from $S^{15} \times^\phi (0, +\infty) \to S^8 \times^a (0, +\infty)$ with fibers S^7_b where a, b are functions of t described earlier and $S^7 = S^7(1)$. Here $S^8 \times^a (0, +\infty)$ denotes the product of $S^8(r)$ warped over $(0, +\infty)$ using $a(t) = \sinh(t)$ for the warping function.

Finally, since ϕ is a function from $(0, +\infty)$ to $[0, +\infty)$, the distance sphere $S^{15}_{a,b}$ where $a = \sinh \phi(t)$ and $b = \sinh \phi(t) \cosh \phi(t)$ is indeed the distance sphere in $\mathbb{O}\mathbf{H}^2$ at distance $\phi(t)$ from the origin. Scaling this metric on $S^{15}_{a,b}$ throughout by the factor $s = \dfrac{\sinh t}{\sinh \phi(t)}$, we get a sphere $S^{15}_{sa,sb}$ where $sa = \sinh t$ and $sb = \sinh t \cosh \phi(t)$. But this is the distance sphere in $S^{15} \times^\phi (0, +\infty)$ at distance t from the origin.

Since S^{15} is a Riemannian hypersurface in $\mathbb{O}\mathbf{H}^2$ and also in $S^{15} \times^\phi (0, +\infty)$, in the foregoing calculations, it is important for us to consider the shape operator $\mathcal{L}$ of $S^{15} \subset \mathbb{O}\mathbf{H}^2$ and the shape operator L of $S^{15} \subset S^{15} \times^\phi (0, +\infty)$ corresponding to the normal vector field N on S^{15}. The shape operator is a linear operator acting on each tangent space T_pS^{15} at $p \in S^{15}$. We shall compute it by its action on vectors X, U belonging to the vertical and horizontal subspaces respectively. Using formulas from O'Neill [20], we get the following for $S^{15} \subset S^{15} \times^{\phi} (0, +\infty)$:

$$L(X) = (\coth t + \tanh(\phi(t))\dot{\phi}(t))X \quad \text{and} \quad L(U) = \coth tU.$$

And for $S^{15} \subset \mathbb{O}\mathbf{H}^2$ we get,

$$\mathcal{L}(X) = (\coth \phi(t) + \tanh \phi(t))X \quad \text{and} \quad \mathcal{L}(U) = \coth \phi(t)U.$$

Calculation of $\overline{K}(u, N)$.

$$\begin{aligned}\overline{K}(u, N) &= \overline{K}(\sin \sigma X + \cos \sigma U, N)\\ &= \sin^2 \sigma \overline{K}(X, N) + \cos^2 \sigma \overline{K}(U, N)\\ &\quad + 2 \sin \sigma \cos \sigma (\overline{R}(U, N)N \cdot X).\end{aligned}$$

Now $X \in \eta_1$ and since the fibers $S^7 \times^{\phi} (0, +\infty)$ are totally geodesic in $S^{15} \times^{\phi} (0, +\infty)$, the vector $\overline{R}(N, X)N \in \eta_1$. Therefore $(\overline{R}(U, N)N \cdot X) = -(\overline{R}(N, U)N \cdot X) = -(\overline{R}(N, X)N \cdot U) = 0$ since $U \in \eta_2$ and η_1 and η_2 are complementary distributions on S^{15}. Since $S^7 \times^{\phi} (0, +\infty)$ are totally geodesic in $S^{15} \times^{\phi} (0, +\infty)$ and since $X \in \eta_1$, the curvature $\overline{K}(X, N)$ in $S^{15} \times^{\phi} (0, +\infty)$ is the same as the curvature of the plane $\{X, N\}$ in $S^7 \times^b (0, +\infty)$. Since the metric on $S^7 \times^b (0, +\infty)$ is $dt^2 + b^2(dS^7)^2$, the curvature of $\{X, N\}$ is $-\frac{\ddot{b}}{b}$ where $b = \sinh t \cosh \phi(t)$. Therefore,

$$\overline{K}(X, N) = -(1 + \tanh \phi(t)\ddot{\phi}(t) + (\dot{\phi}(t))^2 + 2 \coth t \tanh \phi(t)\dot{\phi}(t)).$$

Since U is tangent to S^{15}, $[U, N] = 0$. And since $S^{15}(1) \times^{\phi} (0; +\infty) \to S^8(r) \times^a (0, +\infty)$ is a Riemannian submersion, by O'Neill's submersion formula [20], $\overline{K}(U, N)$ is the curvature of the plane spanned by $\{U, N\}$ in $S^8(r) \times^a (0, +\infty)$. Thus $\overline{K}(U, N) = -\frac{\ddot{a}}{a} = -\frac{\sinh t}{\sinh t} = -1$. Piecing together these components we get,

$$\begin{aligned}\overline{K}(u, N) = -1 - \sin^2 \sigma (\ddot{\phi}(t) \tanh \phi(t) + (\dot{\phi}(t))^2\\ + 2\dot{\phi}(t) \tanh \phi(t) \coth t).\end{aligned} \tag{2}$$

Calculation of $\overline{K}(u,v)$.

Since the distance sphere in $S^{15}\times^{\phi}(0,+\infty)$ is the distance sphere in $\mathbb{O}\mathbf{H}^2$ scaled by $s=\dfrac{\sinh t}{\sinh\phi(t)}$, $\|su\|=\|sv\|=1$ in $\mathbb{O}\mathbf{H}^2$. Using Gauss' equation for the submanifold $S^{15}\subset S^{15}\times^{\phi}(0,+\infty)$,

$$\text{(a)}\qquad \overline{K}(u,v)=K_{S^{15}}(u,v)-((Lu\cdot u)(Lv\cdot v)-(Lu\cdot v)^2)$$

where $K_{S^{15}}(u,v)$ is the curvature of $\{u,v\}$ in $S^{15}\subset S^{15}\times^{\phi}(0,+\infty)$. Similarly, using Gauss' equation for $S^{15}\subset\mathbb{O}\mathbf{H}^2$,

$$\text{(b)}\qquad \begin{aligned}\hat{K}(su,sv)&=\mathcal{K}_{S^{15}}(su,sv)\\ &\qquad-(\langle\mathcal{L}(su),su\rangle\langle\mathcal{L}(sv),sv\rangle-\langle\mathcal{L}(su),sv\rangle^2).\end{aligned}$$

where $\hat{K}$ is the curvature in $\mathbb{O}\mathbf{H}^2$ and $K_{S^{15}}$ is the curvature of $\{su,sv\}$ in $S^{15}\subset\mathbb{O}\mathbf{H}^2$. Since the metrics on the distance spheres in $S^{15}\times^{\phi}(0,+\infty)$ and $\mathbb{O}\mathbf{H}^2$ differ by the scaling factor s, we have

$$\frac{1}{s^2}\mathcal{K}_{S^{15}}(su,sv)=K_{S^{15}}(u,v).$$

Therefore $s^2\times$ (a) $-$ (b) and rearranging gives,

$$\text{(3)}\qquad \begin{aligned}\overline{K}(u,v)&=\frac{1}{s^2}(\langle\mathcal{L}(su),su\rangle\langle\mathcal{L}(sv),sv\rangle-\langle\mathcal{L}(su),sv\rangle^2)\\ &\qquad-((Lu\cdot u)(Lv\cdot v)-(Lu\cdot v)^2)+\frac{1}{s^2}\hat{K}(su,sv).\end{aligned}$$

We now calculate the terms on the right-hand side.

A calculation yields,

$$\begin{aligned}&\frac{1}{s^2}(\langle\mathcal{L}(su),su\rangle\langle\mathcal{L}(sv),sv\rangle-\langle\mathcal{L}(su),sv\rangle^2)\\ &=\frac{1}{s^2}(\coth^2\phi+\sin^2\alpha+\sin^2\sigma+\sin^2\sigma\sin^2\alpha\tanh^2\phi\\ &\quad-\sin^2\sigma\sin^2\alpha(\coth^2\phi+\tanh^2\phi+2)\langle sX,sY\rangle^2\\ &\quad-\cos^2\sigma\cos^2\alpha\coth^2\phi\langle sU,sV\rangle^2\\ &\quad-2\sin\sigma\cos\sigma\sin\alpha\cos\alpha(1+\coth^2\phi)\langle sX,sY\rangle\langle sU,sV\rangle)\end{aligned}$$

and

$$
\begin{aligned}
&((L(u)\cdot u)(L(v)\cdot v) - (L(u)\cdot v)^2)\\
&= \coth^2 t + \tanh^2\phi(\dot\phi)^2 \sin^2\sigma \sin^2\alpha + \coth t \tanh\phi\dot\phi(\sin^2\sigma + \sin^2\alpha)\\
&\quad - \sin^2\sigma\sin^2\alpha(\coth t + \tanh\phi\dot\phi)^2 (X\cdot Y)^2\\
&\quad - \cos^2\sigma\cos^2\alpha\coth^2 t(U\cdot V)^2\\
&\quad - 2\sin\sigma\sin\alpha\cos\sigma\cos\alpha\coth t(\coth t + \tanh\phi\dot\phi)(X\cdot Y)(U\cdot V).
\end{aligned}
$$

Since $(u\cdot v) = 0$, we get, $-\sin\alpha\sin\sigma(X\cdot Y) = \cos\alpha\cos\sigma(U\cdot V)$. Using this identity and the relations $\langle sX, sY\rangle = (X\cdot Y)$, $\langle sU, sV\rangle = (U\cdot V)$, the term

$$
\frac{1}{s^2}(\langle\mathcal{L}(su), su\rangle\langle\mathcal{L}(sv), sv\rangle - \langle\mathcal{L}(su), sv\rangle^2) \\
- ((L(u)\cdot u)(L(v)\cdot v) - (L(u)\cdot v)^2)
$$

simplifies to

$$
\left(\frac{1}{s^2} - 1\right) + (\sin^2\sigma + \sin^2\alpha)\left(\frac{1}{s^2} - \coth t\tanh\phi\dot\phi\right) \\
+ \sin^2\sigma\sin^2\alpha\tanh^2\phi(1-(X\cdot Y)^2)\left(\frac{1}{s^2} - (\dot\phi)^2\right). \tag{4}
$$

Now, to calculate $\hat{K}(su, sv)$ we use the description of the Riemann curvature tensor $\hat{R}$ of the Cayley hyperbolic plane $\mathbb{O}\mathbf{H}^2$ in [4]. However, the action of the representation of Spin(9) in [4] is different from the action described in [17]. Indeed, the map $(x, y) \mapsto (x, \bar{y})$ of $\mathbb{O}\times\mathbb{O} \to \mathbb{O}\times\mathbb{O}$ sends the Spin(9) action in [17] to that in [4]. In particular, the $\mathbb{O}$-lines in Mostow's description of $\mathbb{O}\mathbf{H}^2$ go to $\mathbb{O}$-lines, i.e., 8-dimensional $\mathbb{R}$-subspaces $\mathcal{R}$ of the tangent space to $\mathbb{O}\mathbf{H}^2$ at 0 such that $\hat{K}(\mathcal{P}) = -4$ for each 2-plane $\mathcal{P}\subset\mathcal{R}$, in the description in [4] under the above map. Applying this map and using the formula for sectional curvature $\hat{K}$ in [4] yields,

$$
\hat{K}(su, sv) = -1 - 3\cos^2\theta \tag{5}
$$

where θ is the angle between the vector sv and the unique $\mathbb{O}$-line $\mathbb{O}u$ containing the vector su. Since $\mathbb{O}u$ is an 8-dimensional subspace, it is important to note that

$$
\theta = \min_{\substack{w\in\mathbb{O}u\\ \|w\|=1}} \measuredangle\,(w, sv).
$$

Hence the value $\cos\theta$ is the maximum for all such angles. Putting (4) and (5) into (3) we get

$$(6)\quad \begin{aligned}\overline{K}(u,v) &= \left(\frac{1}{s^2}-1\right)\\ &+(\sin^2\sigma+\sin^2\alpha)\left(\frac{1}{s^2}-\coth t\tanh\phi(t)\cdot\dot{\phi}(t)\right)\\ &+\sin^2\sigma\sin^2\alpha\tanh^2\phi(t)(1-(X\cdot Y)^2)\left(\frac{1}{s^2}-(\dot{\phi}(t))^2\right)\\ &+\frac{1}{s^2}(-1-3\cos^2\theta).\end{aligned}$$

Calculation of $(\overline{R}(u,N)v\cdot u)$.

Using the fact that $u=\sin\sigma X+\cos\sigma U$ and $v=\sin\alpha Y+\cos\alpha V$, we first expand out $(\overline{R}(u,N)v\cdot u)$ into 8 terms.

Claim 1. The terms $(\overline{R}(X,N)Y\cdot U)$, $(\overline{R}(U,N)Y\cdot X)$ and $(\overline{R}(X,N)V\cdot X)$ are all zero.

Proof. The vectors X, Y belong to the vertical subspace tangential to the fiber S^7 of the distance sphere S^{15}. And since $S^7\times^\phi(0,+\infty)$ is totally geodesic in $S^{15}\times^\phi(0,+\infty)$ we conclude that the vectors $\overline{R}(X,N)Y$, $\overline{R}(Y,X)N$ and $\overline{R}(X,N)X$ are tangent to $S^7\times^\phi(0,+\infty)$. Since the vectors U and V belong to the horizontal space, the terms $(\overline{R}(X,N)Y\cdot U)$, $(\overline{R}(Y,X)N\cdot U)$ and $(\overline{R}(X,N)X\cdot V)$ are zero and this completes the proof of Claim 1. q.e.d.

To analyze the remaining terms we use the Codazzi-Mainardi equation for the submanifold S^{15} of $S^{15}\times^\phi(0,+\infty)$. For this we recall the shape operator L acting on the vectors tangent to S^{15}. We have $L(X)=(\coth t+\tanh\phi\dot{\phi})X$ for vectors X in vertical subspace and $L(U)=(\coth t)U$ for vectors U in the horizontal subspace.

For vectors $\alpha,\beta,\gamma\in T_pS^{15}$, the Codazzi-Mainardi equation gives $(\overline{R}(\alpha,\beta)\gamma\cdot N)=-(\mathrm{Tor}_L(\alpha,\beta)\cdot\gamma)$ where $\mathrm{Tor}_L(\alpha,\beta)=\overline{D}_\alpha L(\beta)-\overline{D}_\beta L(\alpha)-L([\alpha,\beta])$ where $\overline{D}$ is the Riemannian connection on $S^{15}\times^\phi(0,+\infty)$. Applying this equation for the remaining 5 terms gives $(\overline{R}(X,N)X\cdot Y)=0$, $(\overline{R}(U,N)U\cdot V)=0$ and $(\overline{R}(U,N)U\cdot Y)=0$. For the remaining two terms we get $(\overline{R}(U,N)V\cdot X)=(\tanh\phi)\dot{\phi}(\overline{D}_VU\cdot X)$ and $(\overline{R}(X,N)V\cdot U)=(\tanh\phi)\dot{\phi}((\overline{D}_VU-\overline{D}_UV)\cdot X)$. Thus $(\overline{R}(u,N)v\cdot u)=\sin\sigma\cos\sigma\cos\alpha(\tanh\phi)\dot{\phi}((2\overline{D}_VU-\overline{D}_UV)\cdot X)$. Since the distance spheres in $S^{15} \times^{\phi} (0,+\infty)$ are gotten by scaling the metric on the distance spheres in $\mathbb{O}\mathbf{H}^2$ by a factor $s = \dfrac{\sinh t}{\sinh \phi(t)}$ they have the same affine connection. We therefore have, for vectors $U, V, X \in T_pS^{15}$, that tangential part of $\overline{D}_V U =$ tangential part of $\hat{D}_V U$ where $\hat{D}$ denotes the Riemannian connection on $\mathbb{O}\mathbf{H}^2$. Hence $(\overline{D}_V U \cdot X) = (\hat{D}_V U \cdot X) = s^2\langle \hat{D}_V U, X\rangle$ where $(\cdot)$ and $\langle\ ,\ \rangle$ denote the Riemannian metrics on $S^{15} \times^{\phi} (0,+\infty)$ and on $\mathbb{O}\mathbf{H}^2$ respectively.

On the other hand, proceeding exactly as above while simplifying $(\overline{R}(u,N)v \cdot u)$, we can show that

$$\langle \hat{R}(su,sN)sv,su\rangle = \sin\sigma\cos\sigma\cos\alpha(\tanh\phi)s^4\langle 2\hat{D}_V U - \hat{D}_U V, X\rangle.$$

Now, using relations $(\overline{D}_V U \cdot X) = s^2(\hat{D}_V U, X\rangle$ and $(\overline{D}_U V \cdot X) = s^2\langle \hat{D}_U V, X\rangle$ deduced above we get the following:

$$(\overline{R}(u,N)v \cdot u) = \frac{\dot{\phi}}{s^2}\langle \hat{R}(su,sN)sv,su\rangle.$$

We then wish to calculate the term on the right-hand side using the formula for the curvature tensor for the Cayley hyperbolic plane $\mathbb{O}\mathbf{H}^2$ described in [4]. To be able to do so we must as before first transform our description of $\mathbb{O}\mathbf{H}^2$ to the description in [4] via the map $f : (x,y) \mapsto (x,\overline{y})$ of $\mathbb{O}^2 \to \mathbb{O}^2$. Also our curvature operator $\hat{R}$ is negative of that in [4]. Making these necessary changes and using the formula for the curvature operator $\hat{R}$ in [4, page 52], a calculation yields,

$$\begin{aligned}\frac{\dot{\phi}}{s^2}\langle \hat{R}(su,sN)sv,su\rangle &= -\frac{3\dot{\phi}}{s}\sin\sigma\cos\sigma\cos\alpha\langle s^2X\overline{U}, s\overline{V}\rangle \\ &= -\frac{3\dot{\phi}}{s}\sin\sigma\cos\sigma\cos\alpha\langle s^2U\overline{X}, sV\rangle.\end{aligned}$$

This together with the fact that

$$\langle s^2u\overline{X}, sv\rangle = \cos\sigma\cos\alpha\langle s^2U\overline{X}, sV\rangle$$

yields

$$(\overline{R}(u,N)v \cdot u) = -\frac{3\dot{\phi}}{s}\sin\sigma\langle s^2u\overline{X}, sv\rangle = -\frac{3\dot{\phi}}{s}\sin\sigma\cos\omega \tag{7}$$

where ω is the angle between the unit length vectors $s^2u\overline{X}$ and sv. Finally, putting together the calculations (2), (6) and (7) into (1) gives,

$$\begin{aligned}
\overline{K}(P) = \cos^2\tau \Bigg(& \frac{1}{s^2} - 1 \\
& + (\sin^2\sigma + \sin^2\alpha)\left(\frac{1}{s^2} - \coth(t)(\tanh\phi(t))\dot{\phi}(t)\right) \\
& + \sin^2\sigma\sin^2\alpha\tanh^2\phi(t)(1-(X\cdot Y)^2)\left(\frac{1}{s^2} - \dot{\phi}(t)^2\right) \\
& + \frac{1}{s^2}(-1-3\cos^2\theta)\Bigg) \\
& + \sin^2\tau(-1-\sin^2\sigma(\ddot{\phi}(t)\tanh\phi(t) + \dot{\phi}(t)^2 \\
& \quad + 2\dot{\phi}(t)(\tanh\phi(t))\coth(t)) \\
& - 6\sin\tau\cos\tau\sin\sigma\frac{\dot{\phi}}{s}\cos\omega.
\end{aligned}$$

Combining and regrouping the above term we get

$$\begin{aligned}
(8)\quad & \overline{K}(P) \\
& = -1 - 3\left(\frac{\cos\tau\cos\omega}{s} + \sin\tau\sin\sigma\dot{\phi}(t)\right)^2 \\
& \quad - \frac{3\cos^2\tau}{s^2}(\cos^2\theta - \cos^2\omega) \\
& \quad + \cos^2\tau(\sin^2\sigma + \sin^2\alpha)\left(\frac{1}{s^2} - \coth(t)(\tanh\phi(t))\dot{\phi}(t)\right) \\
& \quad - \sin^2\tau\sin^2\sigma\ddot{\phi}(t)\tanh\phi(t) \\
& \quad - 2\sin^2\tau\sin^2\sigma\dot{\phi}(t)(\tanh\phi(t)\coth(t) - \dot{\phi}(t)) \\
& \quad + \cos^2\tau\sin^2\sigma\sin^2\alpha\tanh^2\phi(t)(1-(X\cdot Y)^2)\left(\frac{1}{s^2} - \dot{\phi}(t)^2\right).
\end{aligned}$$

We now proceed to choose functions ϕ so that given an $\epsilon > 0$, the curvature $\overline{K}(P)$ satisfies $-4-\epsilon \leq \overline{K}(P) \leq -1+\epsilon$ for all plane sections P in $S^{15} \times^{\phi} (0, +\infty)$.

Following [9] first fix a smooth function $\psi : \mathbb{R} \to [0,1]$ such that

$$\begin{aligned} \dot{\psi}(t) \geq 0 \ \text{ for all } \ t \in [1,2] \\ \psi^{-1}(0) = (-\infty, 1) \ \text{ and} \\ \psi^{-1}(1) = [2, +\infty). \end{aligned}$$

For each $c \geq 1$, let $\phi_c(t) = \psi\left(\frac{\ln t}{c}\right) t$ for all $t > 0$. Therefore $\phi_c(t) = 0$ for $t \in (0, e^c]$ and $\phi_c(t) = t$ for $t \in [e^{2c}, +\infty)$. As in [9] observe that the following limits hold uniformly in t:

$$\text{(9)} \qquad \begin{cases} \lim_{c\to+\infty} |\ddot{\phi}_c(t)| = 0, \\ \limsup_{c\to+\infty} \dot{\phi}_c(t) \leq 1, \\ \limsup_{c\to+\infty} \left(\frac{1}{s^2} - (\dot{\phi})^2\right) \leq 0 \\ \lim_{c\to+\infty} \dot{\phi}_c(t)(\tanh \phi_c(t) \coth t - 1) = 0. \end{cases}$$

(The 3rd inequality is a bit different from the corresponding inequality posited in [9, (2.22)].) Now, for the angles θ and ω as in (8) we have the following:

Lemma 1. $|\cos\omega| \leq \cos\theta$.

Proof. Recall that ω is defined by the relation $\cos\omega = \langle s^2 u\overline{X}, sv\rangle$. Consider the vector $u^X := (s^2 \sin\sigma, s^2\cos\sigma U\overline{X}) = s^2 u\overline{X}$. It is easy to see that $\pm u^X \in \mathbb{O}u$ where $\mathbb{O}u$ is the unique $\mathbb{O}$-line containing the vector su. (Note that $X = -\overline{X}$ and hence $(U\overline{X})X = U(\overline{X}X) = U$.) And obviously $\langle \pm u^X, sv\rangle = \pm\cos\omega$. Since $\theta = \min_{\substack{w\in\mathbb{O}u \\ |w|=1}} \measuredangle(w, sv)$, we conclude that $|\cos\omega| \leq \cos\theta$. q.e.d.

Lemma 1 together with formulas (8) and (9) yield the following result when $\phi(t)$ is one of the functions $\phi_c(t)$.

Lemma 2. *If $t \in (0, e^c]$ and $\phi = \phi_c$, then $\overline{K}(P) = -1$. Moreover, for all $t > 0$, the following limit holds uniformly in t:*

$$\limsup_{c\to+\infty} \overline{K}(P) = -1$$

where $\phi = \phi_c$. Also $S^{15} \times^0 (0, +\infty)$ is $\mathbb{R}\mathbf{H}^{16}$ less a point and $S^{15} \times^t (0, +\infty)$ is $\mathbb{O}\mathbf{H}^2$ less a point. Hence $S^{15} \times^{\phi_c} (0, e^c]$ can be identified *with a closed ball of radius e^c in $\mathbb{R}\mathbf{H}^{16}$ with its center deleted. And $S^{15} \times^{\phi_c} [e^{2c}, +\infty)$ can be identified with $\mathbb{O}\mathbf{H}^2$ from which an open ball of radius e^{2c} is deleted.*

To obtain a lower bound for $\overline{K}(P)$ we need the following lemma.

Lemma 3.

(i) *The maximum value of $|C \cos y \cos z + D \sin y \sin z|$ is $\max\{|C|, |D|\}$, as both y and z vary over $\mathbb{R}$.*

(ii) *The maximum value of*

$$B\cos^2\tau\cos^2\theta + (1-B)\cos^2\tau\frac{a}{3} + 2\sqrt{B}\cos\tau\sin\tau\sin\sigma\cos\omega + \sin^2\tau\sin^2\sigma$$

is 1, where $\tau \in \mathbb{R}$, $B \in [0,1]$, $\alpha, \sigma \in [0, \pi_2]$ and angles θ and ω are as in (8). And $a = \sin^2\sigma + \sin^2\alpha + \sin^2\sigma\sin^2\alpha$.

Proof. We skip the proof of (i) which can be proved by elementary calculus and proceed directly to prove (ii).

(ii) It is convenient to set

$$f = B\cos^2\tau\cos^2\theta + (1-B)\cos^2\tau\frac{a}{3} + 2\sqrt{B}\cos\tau\sin\tau\sin\sigma\cos\omega + \sin^2\tau\sin^2\sigma.$$

We first observe that f is quadratic in $\sin\tau$ and $\cos\tau$. Hence, letting

$$M = \begin{pmatrix} B\cos^2\theta + (1-B)\frac{a}{3} & \sqrt{B}\ \sin\sigma\cos\omega \\ \sqrt{B}\ \sin\sigma\cos\omega & \sin^2\sigma \end{pmatrix}$$

we see that $f = (\cos\tau\ \ \sin\tau) M \begin{pmatrix}\cos\tau \\ \sin\tau\end{pmatrix}$. By a linear algebra argument it follows that $f \le 1$ for all $\tau \in \mathbb{R}$ if and only if the maximum eigenvalue of M is less than or equal to 1; i.e., $f \le 1$ for all $\tau \in \mathbb{R}$ if and only if $g = \text{Trace}(M) - \text{Determinant}(M) \le 1$.

Now

$$g = B\cos^2\theta + (1-B)\frac{a}{3} + \sin^2\sigma - \left(B\cos^2\theta + (1-B)\frac{a}{3}\right)\sin^2\sigma + B\sin^2\sigma\cos^2\omega.$$

Since g is linear in B, thinking of g as a function of B and fixing θ, ω, σ and α, it is easy to see that the maximum value of g occurs at either $B = 0$ or at $B = 1$. Therefore, to prove the lemma, it is sufficient to show that $g\big|_{B=0} \leq 1$ and $g\big|_{B=1} \leq 1$. It is easy to see that $g\big|_{B=0} \leq 1$. To show $g\big|_{B=1} \leq 1$, we show equivalently that for all $\tau \in \mathbb{R}$, $f\big|_{B=1} \leq 1$. This fact follows easily by observing that the formula for $\overline{K}(P)$ in (8) for values of $t \geq e^{2c}$ (in which case $S^{15} \times^{\phi_c} [e^2, +\infty)$ is $\mathbb{O}\mathbf{H}^2$ less an open ball of radius e^{2c}) reduces to $-3f\big|_{B=1} - 1$ from which it follows that $f\big|_{B=1} \leq 1$ for all $\tau \in \mathbb{R}$. q.e.d.

Lemma 4. $\liminf\limits_{c\to+\infty} \overline{K}(P) = -4$.

Proof. It suffices, because of Lemma 2, to show that $\liminf\limits_{c\to+\infty} \overline{K}(P) \geq -4$. Because of (8) and (9), this is equivalent to showing that $\limsup\limits_{c\to+\infty} v \leq 3$, where

$$\begin{aligned} v = 3 &\left(\frac{\cos^2\tau\cos^2\theta}{s^2} + \sin^2\tau\sin^2\sigma(\dot{\phi}_c(t))^2 \right. \\ &\left. + \frac{2}{s}\sin\tau\cos\tau\sin\sigma\cos\omega\dot{\phi}_c(t)\right) \\ &+ \cos^2\tau(\sin^2\sigma + \sin^2\alpha)\left(\coth t\tanh\phi_c(t)\dot{\phi}_c(t) - \frac{1}{s^2}\right) \\ &+ 2\sin^2\tau(\sin^2\sigma)\dot{\phi}_c(t)(\tanh\phi_c(t)\coth t - \dot{\phi}_c(t)) \\ &+ \cos^2\tau\sin^2\sigma\sin^2\alpha\tanh^2\phi_c(t)(1-(X\cdot Y)^2)\left((\dot{\phi}_c(t))^2 - \frac{1}{s^2}\right). \end{aligned}$$

Let $B = \dfrac{1}{s^2}$ and $x = \psi\left(\dfrac{\ln t}{c}\right)$ and define v_1 by

$$\begin{aligned} v_1 = 3&\Big(B\cos^2\tau\cos^2\theta + \sin^2\tau(\sin^2\sigma)x^2 \\ &+ 2\sqrt{B}\sin\tau\cos\tau\sin\sigma(\cos\omega)x\Big) \\ &+ \cos^2\tau(\sin^2\sigma + \sin^2\alpha)(x - B) + 2\sin^2\tau(\sin^2\sigma)x(1-x) \\ &+ \cos^2\tau\sin\sigma\sin^2\alpha\tanh^2\phi_c(t)(1-(X\cdot Y)^2)(x^2 - B). \end{aligned}$$

Using (9), it is easy to see that $v_1 - v$ converges to 0 uniformly as $c \to +\infty$. Therefore, it suffices to show that $\limsup\limits_{c\to+\infty} v_1 \leq 3$. Since the maximum values of $\tanh^2 \phi_c(t)$ and $(1-(X\cdot Y)^2)$ is 1, it suffices to show that $\limsup_{c\to+\infty} v_2 \le 3$ where

$$\begin{aligned} v_2 = 3\Big(&B\cos^2\tau\cos^2\theta + \sin^2\tau(\sin^2\sigma)x^2 \\ &+ 2\sqrt{B}\ \sin\tau\cos\tau\sin\sigma(\cos\omega)x\Big) \\ &+ \cos^2\tau(\sin^2\sigma+\sin^2\alpha)(x-B) + 2\sin^2\tau(\sin^2\sigma)x(1-x) \\ &+ \cos^2\tau\sin^2\sigma\sin^2\alpha(x^2-B). \end{aligned}$$

Now, define

$$\begin{aligned} v_3 = 3\Big(&B\cos^2\tau\cos^2\theta + \sin^2\tau\sin^2\sigma + 2\sqrt{B}\ \sin\tau\cos\tau\sin\sigma\cos\omega\Big) \\ &+ \cos^2\tau(\sin^2\sigma+\sin^2\alpha+\sin^2\sigma\sin^2\alpha)(1-B). \end{aligned}$$

Since v_2 is a continuous function of B, τ, σ, α, θ, ω and x, we have $\lim_{x\to1}(v_2-v_3)=0$ uniformly in $B\in[0,1]$, $\tau,\omega\in\mathbb{R}$ and $\sigma,\alpha,\theta\in[0,\pi/2]$.

Since $v_3 \le 3$ by Lemma 3(ii), it follows therefore that, given an $\epsilon>0$, there exists a $\delta>0$ such that $x>1-\delta$ implies $v_2\le 3+\epsilon$. Hence to complete the proof of the Lemma, we may assume that $x\le 1-\delta$. This, together with the fact that $\phi_c(t)=xt$ and the specific choice of functions $\phi_c(t)$ can be used to show that

$$\lim_{c\to+\infty} B = 0 \quad \text{uniformly in } \ t. \tag{10}$$

Now (10), $x\le1-\delta$ and Lemma 3(i) together imply that $\limsup_{c\to+\infty} v_2\le 3$ which completes the proof of the Lemma 4. q.e.d.

3. Detecting exotic smooth structures

The purpose of this section is to prove the following result.

Theorem. *Let M^{16} be any closed locally Cayley hyperbolic manifold. Given $\epsilon>0$ then there exists a finite sheeted cover $\mathcal{N}^{16}$ of M^{16} such that the following is true for any finite sheeted cover N^{16} of $\mathcal{N}^{16}$.*

(a) *N^{16} is not diffeomorphic to $N^{16}\#\Sigma^{16}$.*

(b) *$N^{16}\#\Sigma^{16}$ supports a negatively curved Riemannian metric whose sectional curvatures are contained in the interval $[-4-\epsilon,-1]$.*

Here Σ^{16} is the unique closed, oriented smooth 16-dimensional manifold which is homeomorphic but not diffeomorphic to the sphere S^{16}. The existence and uniqueness of Σ^{16} is a consequence of the following result which is implicit in [14]. However, for the reader's convenience, we derive it here from the Sullivan-Wall surgery exact sequence.

Proposition. *The group of smooth homotopy spheres θ_{16} is cyclic of order 2.*

Proof. We have the following surgery exact sequence from [23]:

$$0 \to \theta_{16} \to \pi_{16}(F/O) \to L_{16}(O) = \mathbb{Z}.$$

This sequence together with the fact that θ_{16} is a finite group show that θ_{16} can be identified with the subgroup S of $\pi_{16}(F/O)$ consisting of all elements having finite order. Next consider the exact sequence

$$\pi_{16}(O) \xrightarrow{J} \pi_{16}(F) \to \pi_{16}(F/O) \to \pi_{15}(O) = \mathbb{Z}.$$

This sequence and the fact that $\pi_{16}(F) = \pi_{16}^s$ is a finite group show that S can be identified with cokernel of J. Recall now that Adams [1] proved that J is monic. This result together with the facts that $\pi_{16}(O) = \mathbb{Z}_2$ and $\pi_{16}^s = \mathbb{Z}_2 \oplus \mathbb{Z}_2$ (cf. [22]) show that $\theta_{16} = \mathbb{Z}_2$. q.e.d.

Now, the proof the theorem posited in the beginning of this section follows the pattern established in [8] and [9]. The main result of Okun's thesis [21, Theorem 5.1] gives a finite sheeted cover $\mathcal{N}^{16}$ of M^{16} and a tangential map $f : \mathcal{N}^{16} \to \mathbb{O}\mathbf{P}^2$. And we can arrange that $\mathcal{N}^{16}$ has arbitrarily large preassigned injectivity radius r by taking larger covers since $\pi_1(M^{16})$ is residually finite. Once r is determined, then this is the manifold $\mathcal{N}^{16}$ posited in the theorem. The argument in [9, pp. 69–70] is now easily adapted to yield the following lemma since the other ingredients — Mostow's strong rigidity theorem [17], its topological analogue [7] and Kirby-Siebenmann smoothing theory [14, pp. 25 and 194] — remain valid.

Lemma 0. *Let N^{16} be any finite sheeted cover of $\mathcal{N}^{16}$. If $N^{16}\#\Sigma^{16}$ is diffeomorphic to N^{16}, then $\mathbb{O}\mathbf{P}^2\#\Sigma^{16}$ is concordant to $\mathbb{O}\mathbf{P}^2$.*

The octave projective plane $\mathbb{O}\mathbf{P}^2$ is the mapping cone of the Hopf map $p : S^{15} \to S^8$. Let $\phi : \mathbb{O}\mathbf{P}^2 \to S^{16}$ be the collapsing map obtained by identifying S^{16} with $\mathbb{O}\mathbf{P}^2/S^8$ in an orientation preserving way.

Lemma 1. *The homomorphism $\phi^* : [S^{16}, \mathrm{Top}/O] \to [\mathbb{O}\mathbf{P}^2, \mathrm{Top}/O]$ is monic.*

Proof. Recall that $[X, \mathrm{Top}/O]$ is the zeroth cohomology group of X in an extraordinary cohomology theory. By considering the long exact sequence in this theory determined by the pair $(\mathbb{O}\mathbf{P}^2, S^8)$ and using the identification of $\mathbb{O}\mathbf{P}^2$ with the mapping cone of ϕ, it is seen that ϕ^* is monic if and only if

$$(\Sigma p)^* : [S^9, \mathrm{Top}/O] \to [S^{16}, \mathrm{Top}/O]$$

is the zero homomorphism. Here

$$\Sigma p : S^{16} \to S^9$$

denotes the suspension of p.

To show that $(\Sigma p)^*$ is the zero homomorphism consider the following commutative ladder of groups and homomorphisms:

$$\begin{array}{ccccccc}
\theta_{16} = [S^{16}, \mathrm{Top}/O] & \xrightarrow{\alpha} & [S^{16}, F/O] & \longleftarrow & [S^{16}, F] & \xleftarrow{J} & [S^{16}, O] \\
\uparrow{\scriptstyle (\Sigma p)^*} & & \uparrow{\scriptstyle (\Sigma p)^*} & & \uparrow{\scriptstyle (\Sigma p)^*} & & \\
\theta_9 = [S^9, \mathrm{Top}/O] & \longrightarrow & [S^9, F/O] & \xleftarrow[\beta]{} & [S^9, F]\ . & &
\end{array}$$

The horizontal homomorphisms in this ladder are induced by the natural maps $\mathrm{Top}/O \to F/O$, $F \to F/O$, and $O \to F$. Now the following three facts used in conjunction with a simple "diagram chase" show that $(\Sigma p)^*$ is the zero homomorphism thus proving Lemma 1. q.e.d.

Fact 1. α is monic.

Fact 2. β is an epimorphism.

Fact 3. Image $(\Sigma p)^* \subseteq$ Image J where $(\Sigma p)^* : [S^9, F] \to [S^{16}, F]$ and $J : [S^{16}, O] \to [S^{16}, F]$ is the classical J-homomorphism.

It remains to verify these Facts. Fact 1 is due to Kervaire and Milnor [14]. A more modern proof is given by observing that α is a homomorphism in Sullivan's surgery exact sequence

$$\cdots \longrightarrow L_{17}(0) \longrightarrow \theta_{16} \xrightarrow{\alpha} \pi_{16}(F/O) \longrightarrow \ldots$$

and that $L_{17}(0) = 0$. Fact 2 is due to Adams [1] who showed that

$$J : \pi_8(O) \to \pi_8(F)$$

is monic. (Now consider the homotopy exact sequence for the fibration $O \to F \to F/O$.) Fact 3 is a more special result which we proceed to prove.

During this proof we will use Toda's notation [22, p. 189] for special elements in the stable stems $G_n = \pi_n^s$. Recall that G equal the direct sum of the G_n is an anti-commutative graded ring with respect to composition as multiplication. First note that the homotopy class

$$[\Sigma p] = a\sigma + x \in \pi_7^s$$

where $a \in \mathbb{Z}$ and $x \in \pi_7^s$ has odd order. Using this together with the fact that π_{16}^s has order 4, we see that Image $(\Sigma p)^*$ is generated by the following three elements:

$$v^3 \circ a\sigma, \quad \mu \circ a\sigma, \quad \eta \circ \epsilon \circ a\sigma.$$

And Theorem 14.1 (ii, iii) [22, p. 190] yields that

$$\begin{aligned} v^3 \circ a\sigma &= a(v^2 \circ (v \circ \sigma)) = 0, \\ \eta \circ \epsilon \circ a\sigma &= a(\eta \circ (\sigma \circ \epsilon)) = 0, \quad \text{and} \\ \mu \circ a\sigma &= a(\mu \circ \sigma) = -a(\sigma \circ \mu) = -a(\eta \circ \rho) = a(\rho \circ \eta). \end{aligned}$$

Consequently, Image $(\Sigma p)^*$ is contained in the subgroup of π_{16}^s generated by $\rho \circ \eta$. Hence in order to complete the verification of Fact 3 it suffices to show that

$$\rho \circ \eta \in \text{Image } J.$$

To do this let $\eta_{15} : S^{16} \to S^{15}$ represent the element $\eta \in \pi_1^s$ and notice that the following digram commutes:

$$\begin{array}{ccc} [S^{15}, F] & \xrightarrow{\eta_{15}^*} & [S^{16}, F] \\ {\scriptstyle J'}\uparrow & & \uparrow{\scriptstyle J} \\ [S^{15}, O] & \xrightarrow[\eta_{15}^*]{} & [S^{16}, O] \end{array}$$

where J' denotes the J-homomorphism in dimension 15. We recall that Kervaire and Milnor showed (cf. [18, p. 284]) that Image J' is a cyclic group of order 480. Using this fact together with Toda's calculation of π_{15}^s in [22, p. 189] it is easily seen that

$$\text{either} \quad \rho \in \text{Image } J' \quad \text{or} \quad \rho + \eta \circ k \in \text{Image } J'.$$

Consequently the above commutative diagram shows that

$$\text{either} \quad \rho \circ \eta \in \text{Image } J \quad \text{or} \quad \rho \circ \eta + \eta \circ k \circ \eta \in \text{Image } J.$$

But Theorem 14.1(i) in [22, p. 190] yields that

$$\eta \circ k \circ \eta = \eta^2 \circ k = 0$$

and consequently $\rho \circ \eta \in \text{Image } J$.

But Lemma 1 implies that $\mathbb{O}\mathbf{P}^2 \# \Sigma^{16}$ is *not* concordant to $\mathbb{O}\mathbf{P}^2$ since the concordance classes of smooth structures on a smooth manifold X are in bijective correspondence with $[X, \text{Top}/O]$ provided dim $X > 4$. Thus assertion (a) of the Theorem is a direct consequence of Lemmas 0 and 1.

Now combining the construction of the previous section with [9, Lemma 3.18] it is seen that there exists a number $r_{16} > 0$ (independent of M^{16}) such that if the injectivity radius of $\mathcal{N}^{16}$ is chosen to be larger than r_{16}, then assertion (b) of the Theorem is also true. Since Borel [3] has constructed closed Riemannian manifolds M^{16} whose universal cover $\widetilde{M}^{16}$ is $\mathbb{O}\mathbf{H}^2$, Theorem produces the examples claimed in the Introduction.

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